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IMPOSSIBLE PROBLEMS

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An eight-inch checkerboard can be covered with 32 one-by-two-inch dominos in thousands of different ways. If two opposite corners of the board are removed, 31 dominoes should be sufficient to cover the remainder. But the first attempt to do so ends in failure. So does the second. Everything proceeds smoothly most of the time; minor difficulties arise, but they are met successfully until the end. Then the would-be coverer finds himself with one domino in hand and two non-adjacent squares staring up at him from the board.

A great many people will react to this challenge in quite a predictable fashion. For a few trials they appear confident, laughing in mild embarrassment at failures, and returning to the fray with undiminished courage. Then confidence gives way to grim determination, which gradually changes to frustration. They mutter under their breath about how close they were that time, and how the next try will surely bring success. They act as if they were really trying a different technique each time, whereas the impassionate observer can perceive no change in the approach. Their self-confidence remains so strong that if someone else knocks the board onto the floor before they do, he will be accused of interrupting what would certainly have been the correct solution but is now lost to mankind forever. Such people do not give up; when they stop, it is with the promise that they will do the job easily some other day when they are not so nervous.

A few people take up an entirely different approach after the first few attempts. They try to discern just what goes wrong each time, and if possible to abstract from their trials certain principles that apply *no matter what* steps are taken. Such a careful analyst notices quite early that the two squares left over at the end are always of the same color. Of course the color has nothing to do with the problem, but it provides a clue to the solution. It prompts a reappraisal of the situation, which reveals that the mutilated board has thirty-two squares of one color and only thirty of the other. *No matter where* a domino is placed, it rests on two squares of different colors. So of course thirty-one of them cannot cover this board.

The proof that the problem is impossible is a far cry from a mere admission that it is difficult, or even that an individual considers himself defeated by it. Many people would like to believe that the problem of ten discs and three pegs is impossible, simply because they have not succeeded in solving it! The unequal discs are stacked on one peg in order of size, with the largest on the bottom. They are to be moved to another peg, one at a time, with no disc ever placed atop one smaller than itself. The fact that it takes 1023 moves (provided no errors are made) is discouraging, but has none of the finality of a proof of impossibility. The mathematician who proves the checkerboard uncoverable, or anyone who follows his demonstration, will certainly waste no time in further attempts at a solution.

The mathematician's habit of generalization may tempt him to continue work on a problem even after he proves it impossible in its original form. He can prove to himself, or anyone else who will listen, that a board with two squares of different colors removed can always be covered with dominoes. This is not done by showing successful coverings for each possible mutilation, any more than the original problem was shown impossible by enumerating failures. There is a technique that will guarantee coverage *no matter which* squares are removed. On the other hand, it is easy to demonstrate that four squares, two of each color, can be removed in such a way as to make coverage impossible.

The prover and generalizer is here called a mathematician for want of a better term. Certainly there are some whose qualifications as mathematicians under certain definitions are not in harmony with their ability to solve such problems. Is it fair even to call this a mathematical problem? It is, and this because of the tremendous scope of the word 'mathematics.' If the same word can be used to describe arithmetic and projective geometry, it is certainly versatile enough to include checkerboards and dominoes. One of the things that gives mathematics its great breadth is its willingness to adopt orphaned problems such as this one. The origin of a puzzle may be in physics, philosophy, or fantasy. But when its essential features are abstracted from its outward form, it belongs properly to mathematics. Since, in fact, mathematics is the custodian of **proof,** all impossible problems belong to mathematics, and look to that collection of disciplines for proof of impossibility.

One famous impossible problem had only one year between its presentation by Sam Loyd in 1878 and its published solution in the *American Journal of Mathematics.* The puzzle remained popular, partly because it has capabilities for amusement besides its impossible aspect, but mostly because its impossibility has not been widely recognized. It is not nearly so demonstrable as the colorful analysis of the checkerboard.

This is the "15 puzzle"; it takes many forms, but always has the same basic structure. Fifteen small squares are arranged in a square box that has room for sixteen. The empty space can be filled by sliding an adjacent square into it, and repeated applications of this procedure can give more than ten billion different patterns of numbers, letters, or whatever may be printed on the squares. But there are ten billion other arrangements that are all impossible, and the challenge offered is to arrive at one of them. An imaginative version calls for arranging fifteen letters to spell RATE YOUR MIND PAL. This happens to be impossible or possible, depending on which R is used in which place. An unsuspecting "pal" is shown the correct arrangement, and then handed the puzzle with the letters rearranged so that the wrong R is already in the upper left corner. By leaving it there, he restricts himself to the impossible version.

The most interesting period in the history of an impossible problem is the time before its impossibility is established. When anyone is not aware of the proof, that period is extended for him as an individual. And there are usually enough such people that an impossible problem remains popular indefinitely. An example is the problem of the utilities. Three points on a piece of paper represent sources of Gas, Water, and Electricity; three more points represent customers; it does not matter where the points are placed. The problem is to connect each utility to each customer without allowing any of the nine lines to cross. The problem actually predates the popular use of electricity! Older versions use some other situation as the framework of the same problem.

The proof that such connection is impossible is comparable in difficulty to two or three attempts at a solution. It is effected by consideration of things that happen *no matter where* the lines are drawn. A "solution" is sometimes presented with one line running under one of the houses, which makes the problem quite possible and utterly uninteresting. Such a "cheat" is comparable to cutting a domino in half to finish covering a checkerboard. It detracts from the real nature of the problem, and is not nearly so worthwhile as a proof of impossibility.

Problems such as this one, during their periods of insolubility, goaded mathematicians into a type of thinking that culminated in the development of a new field within mathematics, called "topology." Among the problems now classified as topological are all that call for connecting objects or tracing paths with one continuous curve. The prototype of this class was "The Seven Bridges of Königsberg." Leonhard Euler's proof of the impossibility of walking over the seven bridges in that city once each was a cornerstone in the development of topology. Not content to try a solution, as everyone else in Switzerland was apparently doing in 1735, nor to try to enumerate the unsuccessful methods, Euler proved succinctly that it could not be done. In the true mathematical tradition, he generalized his result to any number and arrangement of bridges, giving the general characteristics that would lead to insolubility, easy solubility, or solubility within certain restrictions.[1]

The three most celebrated impossible problems challenged the world for thousands of years, and were finally defeated (that is, proved impossible) at the end of the nineteenth century. This exceedingly long period of doubt was immensely fruitful for mathematics. Whereas topology developed from certain proofs of impossibility, these three problems led to many developments that ripened before the problems were proved impossible. New approaches and abstractions were forged into a growing battery of weapons for repeated attacks on the problems. By the time the proofs were complete, the by-products of the struggle far over-

[1] Leonhard Euler, "The Seven Bridges of Königsberg," *The World of Mathematics,* 4 volumes (New York: Simon and Schuster, 1956), Vol. I, pp. 573–580.

shadowed the direct accomplishment (or non-accomplishment) of the objectives: squaring a circle, trisecting an angle, or duplicating a cube.

It is greatly to the credit of early mathematicians that they gave serious consideration to these problems. It has "always" been known that they can be solved approximately. The earliest of mathematical manuscripts, dating back at least to 1700 B.C., points out that a square whose side is eight ninths of a circle's diameter has an area quite close to that of the circle. A five-inch cube duplicates a four-inch cube with an error under three per cent. It is a trivial problem to draw two rays that come within a millionth of a degree of trisecting an angle.

The difference between an approximate solution and an exact one is not easy to appreciate. It is no problem at all to solve any one of these problems with an accuracy far beyond that of a physical straightedge and compass in the hands of a human being. But Euclid and others even earlier recognized the validity of idealizing these instruments and the figures they draw. Thus the mind could go beyond the limitations of the brain, and recognize that none of the myriad schemes offered through the ages would *exactly* accomplish the desired objectives.

It is within the realm of geometry to see that two of these problems are inseparably bound up with determining values of the numbers π and $\sqrt[3]{2}$. It is not so easy to see that trisecting an angle is equivalent to extracting a cube root. A decisive step toward the proof of these impossible problems was the development of the correspondence between geometry and algebra by Descartes. The discipline of analytic geometry enables a mathematician to attack the most difficult geometry problems with algebra, and *vice versa*.

Descartes recognized that a line and a circle correspond to certain types of equations, and that the constructions of geometry are thus equivalent to solving simultaneous equations. Strangely enough, a line and a circle that do not intersect have equations that can be solved. But they always involve the number that gives -1 when multiplied by itself, called i. In Descartes' time, even negative numbers were regarded with suspicion, and he called solutions that involved them "false." What, then, could he call such a monstrosity as an intersection of two non-intersecting curves? He used the word "imaginary," a good choice under the circumstances, but one that resulted in an outstanding mathematical misnomer. The numbers typified by i are still called imaginary, even though they have demonstrated their practicality by repeated application to problems that would be impossible without them.

The apparent digression from the original problem into consideration of i proved very important. When a mathematician finds himself in pursuit of a will-o'-the-wisp, it behooves him not to concentrate too rigidly on one goal. He must investigate whatever areas the chase brings him into. The Providence that governs mathematical discovery has exploited that principle to man's great benefit. It has often dangled before him some impossible problem, and kept it out of his grasp until he had ample opportunity to realize that even an unsuccessful search can be rewarding.

This principle should certainly be kept in mind today, when so many great problems are "on the threshold" of solution. Some of those thresholds will never be crossed, or will prove to lead to great disappointment. Consider how long the alchemists were on the threshold of changing lead to gold. Most of them died believing they were closer to their goal than when they started. But it has turned out that only the by-products of their efforts were of value, and the value realized only after a change of emphasis.

The proven value of digression must not be taken as an excuse for wandering from the point of the present discussion, however. A great step forward in the analysis of squaring the circle was the proof that π is not rational, that is, cannot be the quotient of two integers. Many approximations of increasing accuracy were discovered (3, 22/7, 377/120, 355/113) before someone finally showed that such a number would never be exact. The irrationality of π, however, did not settle the problem. It had been shown much earlier that $\sqrt{2}$ is irrational, but a line segment corresponding to that number is very easily constructed. Further investigation turned up new kinds of numbers that would escape the attention of all but the most careful observers. The rational numbers (fractions) seem to fill in the gaps between integers so thoroughly that many people are still incredulous about there being far more irrational numbers than rational. And only those who appreciated this distinction could approach the next step, which was to show that there are numbers that are worse than irrational. Some numbers (including all the rational numbers) can be roots of algebraic equations with integral coefficients, but many more cannot. The demonstration that π is one of the latter (called transcendental numbers) finally settled the argument about squaring the circle. But this stroke was not accomplished until after Napier's invention of logarithms brought to light another now-famous transcendental number, known as e. The climax of the whole development was the discovery that four of the most interesting numbers known to man are intimately related:

$$e^{\pi i} = -1.$$

By the time this discovery was made, all the pieces of the puzzle were ready for final assembly. No one mathematician is given credit for finishing the problem. Even some of the steps were composite works of some of the greatest mathematicians, and they had already been found applicable to many other problems than the impossible ones that helped to stimulate their development. When it was realized that π is transcendental, the squaring of the circle was finally proved impossible, and the other two problems fell along with it. The investigation had produced a full characterization of possibility and impossibility in geometry, settling all such problems for all time. Only certain types of algebraic numbers lend themselves to geometric construction. The analysis exposed some hitherto unsuspected possibilities as well as proving impossibilities. It was shown to be within geometrical rules, for example, to construct a regular polygon of 65,537 sides, and a book is now available giving specific instructions on how to do so.

Some of the problems are still taken up as challenges. An occasional geometry student rebels at the idea that an angle cannot be trisected, and works out a method. It is wrong, of course, but may show ingenuity in the nature of the error. It is possible, for example, to trisect an angle with compass and straightedge if granted only one non-Euclidean privilege—to use both at once. The postulates describe the results of using either instrument alone, and on the basis of these postulates the problem is impossible. And yet if the compass is held alongside the ruler and the two manipulated together for only one step, the trisection can be dispatched without difficulty. It is noteworthy that even such a "cheat" as this will not make it possible to square a circle.

But all of these issues lack importance now. These problems have passed through the doubtful stage, and will never again have the compelling interest they once enjoyed.

A problem of considerably greater scope has been conquered in recent times, and turned out to be more of a disappointment than a triumph. For in this case the thing proved impossible was something mathematicians were trying to *do* rather than to prove incapable of being done. The three geometry problems were long suspected of being impossible. In 1775 the Paris Academy passed a resolution that no more solutions were to be examined of the problems of perpetual motion, duplication of the cube, trisection of the angle, or quadrature of the circle. The men who perceived that such examinations wholly wasted the time of Academy personnel did not live to see their conviction borne out, but history must note that they were right.

The recent problem, however, was one that fine mathematicians, rather than crackpots, were "on the threshold of solving." How disappointing that they should find the threshold lacking another side! Kurt Gödel showed them so in a treatise published in 1931 under the formidable title, "On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems." The young (25) mathematician was challenging Alfred North Whitehead and Bertrand Russell in their own field. They had attempted in their three-volume *Principia* to reduce all of mathematics and logic to a formal system, based on axioms that left nothing unsaid, and from which everything could be proved.

Success in such a venture would have led to a revolution in the teaching of mathematics. Intuition would have been banned from the classroom, and teachers and students compelled to say what they mean without taking things for granted. There would have been tremendous strides in cybernetics. If computing machines could be told everything about mathematics in a few axioms, they could come frighteningly close to human thought, at least in mathematics. Ultimately, mathematics might have been filed away as a subject whose possibilities had been exhausted. It would remain a useful tool, but no longer be a field for imaginative research.

Strangely enough, all of these ventures are being carried forward with vigor. Gödel's proof is well known now, but zealots in some fields apparently have not paused to consider its implications.

Their enthusiasm for their causes seems to have blinded them to the fact that the essential idea has been exploded. It is time to regroup forces, and to seek to salvage whatever is valuable among the by-products of these quests. Such efforts will undoubtedly be fruitful, but the original idea of reducing mathematics to a finite set of axioms is impossible.

It is not easy to describe exactly what Gödel proved, or how, but the simpler problems considered in the preceding pages should give something of a background. It should be clear that the impossibility is only within certain restrictions, and Gödel had to explain just what is wrong with an inadequate set of axioms. He did not take up specific examples, but demonstrated ably what would happen *no matter what* axioms were selected. He succeeded in showing that no set could be complete and at the same time provably consistent. That is, the axioms would fail to imply some truth about arithmetic, or would contradict each other, or, if consistent, could not be proved so within the axiomatic structure itself, which was supposed to have included all of mathematics and logic. It is possible, within these restrictions, to set up axioms that agree with each other and encompass *most* of arithmetic (whatever that may mean). But if more axioms are added to include facts originally omitted, the threat of inconsistency within the axioms grows. The two goals of consistency and completeness cannot be achieved in the same system.

How did Gödel ever prove all this? A popular explanation of a little over 100 pages is available, with footnotes and appendices that allow the reader to select his own level of detail.[2] The book does not contain the full proof, but shows how its author accomplished consideration of what would happen *no matter what* the axioms were. A cleverly devised system identifies every entity or statement of or about arithmetic with a unique number. It is as if Gödel had been given the problem of covering an eight-inch square (not a checkerboard) with its corners removed, and had proceeded to color it himself. And of course his effort was in a field far more extensive than any number of checkerboards. The Gödel numbers are manipulated arithmetically, and shown to retain certain patterns through any proof. Then it can be shown that a consistent set of axioms must leave at least one

[2] Ernest Nagel and James R. Newman, *Gödel's Proof* (New York University Press, 1958).