THE LAST JUDGMENT AND THE LIBERATION OF MATHEMATICS

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In the year 1758 a series of five rather esoteric books were published in London. They dealt with topics such as heaven and hell, spirits from other planets, and an internal meaning in certain books of the Bible. All were in Latin, the international language of scholarship throughout Europe, and the anonymous author was clearly a person of extraordinary learning. Also they all made copious references to the massive eight volume work called *Arcana Coelestia* which had appeared in bookstores, volume after volume, during the previous decade. These books of profound biblical exegesis were also published anonymously, and obviously the more recent five were by the same exceptional author. People who knew of these books and who were also acquainted with Emanuel Swedenborg, the prominent Swedish scientist and philosopher, had little difficulty in coming to the correct conclusion that he was the unknown author.

In this paper I will be concerned primarily with one of the 1758 books—the one whose title, translated into English, is *The Last Judgment and Babylon Destroyed.* (I will refer to this book as *LJ*.) In this small volume Swedenborg calmly makes the amazing claim that the event that all Christianity had been awaiting for centuries, the cataclysmic, world-ending last judgment, had been totally misunderstood. Throughout all of his theological works he had consistently stressed the fact, which for him was repeatedly verified by his own (spiritual) senses, that the spiritual realm is real and substantial. The natural world, by comparison, is if anything *less* real than the spiritual, and the deeper meaning conveyed in sacred scripture pertains to that more real realm of spirit. Thus the biblical prophecies of a last judgment, in order to be correctly understood, must be understood in the spiritual sense rather than the literal natural sense. So now, in his book

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about the last judgment, he explains that this prophesied event refers to a complete reorganization of the spiritual world that had already been accomplished by the Lord during the previous year of 1757.

A few years later, in 1763, Swedenborg published another small booklet of twenty-eight pages called *Continuation Concerning the Last Judgment*, which I shall abbreviate as *CLJ*. This one he published in Amsterdam, but it was still anonymous. It is called *Supplements* in the New Century Edition being published by the Swedenborg Foundation.

It is not my purpose here to attempt to give any sort of analysis of Swedenborg's description of the last judgment or to review the contents of his remarkable reports. At this point I wish simply to consider briefly a few of his comments about the worldly effects of this major spiritual event. The primary goal of the paper is to indicate how these effects may well have included a major change in the nature of mathematics—a change that can be viewed as a liberation. Some of his remarks can be paraphrased as follows:

- The duration of the last judgment, or at least of his observation of it, coincided closely with the calendar year 1757. (See *LJ* 45)
- The last judgment did not occur in the natural world but only in the spiritual world. (See *LJ* 29)
- There will be no immediate change in worldly conditions or even in the external condition of the church. But the internal nature of the church will be changed dramatically to give people greater freedom in thinking about spiritual matters. (See *LJ* 73)
- Before the last judgment people on earth could not be enlightened because of disorders in the spiritual realm which blocked the influx of truth in a way similar to the blocking of sunlight by dense clouds. After the last judgment, communication between heaven and earth was restored. (See *CLJ* 11, 13)

World history of the eighteenth and nineteenth centuries is filled with political and social changes that have extended human spiritual, intellectual, and personal freedom in such profound ways that it is not difficult to attribute them to effects stemming from the great spiritual event of 1757. We need not dwell on obvious examples such as the movement begun in 1776 toward the founding of a democratic nation and the painfully slow process of abolishing slavery. My purpose here is to show that even in the discipline of mathematics, unemotional and abstract as it may be, there were occurring at the same time some profound developments that can also be viewed as liberating. In my opinion they too can be included in the category of events resulting from the last judgment. In support of this view, I will look at the state of mathematics in the middle of the eighteenth century and trace the development of certain ideas that led to the gradual liberation of mathematics.

Mathematics in 1757

The eighteenth century, even before the year of the last judgment, was a period of great mathematical progress. Mathematicians were keeping very busy and making unprecedented progress in applying the methods of calculus, which had been developed by Newton and Leibniz several decades previously, to all manner of physical problems. The applications ranged from basic engineering to astronomy. This kind of mathematics, exploiting the brilliant techniques of the relatively new calculus, was known as analysis, and it was by far the most active and most notable aspect of the mathematics of that day. But it was characterized by its close connection to the physical world and its rather deplorable (by today's standards) lack of rigor. The rigor was destined to be supplied in the coming post-judgment decades, which were also to bring about the severing of the bonds that tied mathematics to the physical world.

The pre-eminent mathematician in 1757, and for many years before and after, was Leonhard Euler (1707–1783)—a native of Switzerland who spent nearly all of his adult professional life at the scientific academies in two foreign cities: St. Petersburg and Berlin. In 1757 he was just fifty years old and at the peak of his power. In this current year of 2007, when Swedenborgians are commemorating the 250th anniversary of the last judgment, mathematicians all over the world are celebrating the 300th anniversary of Euler's birth. The Mathematical Association of America, for example, is publishing this year not one but *five* books about him and his mathematical contributions. Euler is known as the most prolific of mathematicians. His mathematical writings were so widespread and so voluminous that the job of collecting them all and getting them properly edited and published is still going on today, some 224 years after his death. The number of volumes of his collected works is now seventy-seven and still growing.

As great and as prolific as Euler was, he does not play a major role in the account I will give here of what I have called the liberation of mathematics. I have mentioned him primarily to show that in 1757 mathematics was a burgeoning enterprise and to point out the interesting coincidence that this foremost mathematician had his fiftieth birthday in the year of the last judgment. The story that I intend to tell involves the efforts of several other mathematicians who were men of great genius even though their reputations did not match that of Euler. These efforts were made not in the ascendant realm of analysis but rather in the relatively quiet areas of geometry and algebra. In 1757 each of these two branches of mathematics was struggling with its own long-standing problem which seemed to defy solution. In each case the following years would bring the unexpected result that the problem was in fact insoluble, thus bringing to a close an era in the history of each subject. And in each case the negative result, which seemed disappointing at the time, actually led to the opening of a new era of far greater opportunity and potential for advancement than could have been dreamed of previously. In this way both geometry and algebra were liberated from previous narrow restrictions, and in due course the liberation was extended to all of mathematics. The fact that the origins of this liberation occurred in the late eighteenth century suggests that it was a result of the increasing intellectual freedom stemming from the great spiritual reorganization of 1757.

Geometry

First I will consider the case of geometry. During the eighteenth century, geometry was nearly synonymous with the work of Euclid, whose famous book *Elements* (written some 300 years B.C.) had been the definitive text on the subject for millennia. Of course Euclid's work had been lost to European culture during the long centuries of the dark ages, but it had been rediscovered and restored to its preeminent position during the Renaissance. In *Elements*, geometry was presented as a unified body of knowledge, all of which was derived by strict logical reasoning from a set of supposedly self-evident postulates. It was without doubt a masterpiece of human thought, and it was avidly studied not only by mathematicians but by all scholars who wished to become proficient at careful reasoning. We who are living today in this era of constant change can hardly imagine the long-term status that was enjoyed by Euclid. Even in the English-speaking world his presentation of geometry remained standard through the end of the nineteenth century, some 2,300 years after it was first composed. The problem in geometry that was mentioned above concerned the postulates that Euclid had chosen as the basis for his derivation of geometrical knowledge. There were five of them:

1. A straight line may be drawn connecting any two given points.
2. A straight line may be extended continuously in a straight line in either direction.
3. A circle may be drawn with any given point as center and passing through any given second point.
4. All right angles are equal to one another.
5. If a straight line falling on two straight lines makes the interior angles on the same side together less than two right angles, the two straight lines, if extended indefinitely, meet on that side on which the angles are together less than two right angles.

Looking at these postulates, one notices immediately that the fifth one is of a decidedly different character than the other four. It is much longer, more complicated, and lacking in the self-evidence of the others. Why would the great Euclid have chosen it? Would it not be more appropriate to have this kind of statement appear as a derived theorem rather than a postulate? It would have made an excellent theorem, but as a postulate it was just plain ugly. And in mathematics esthetics is an important consideration.

So despite the great respect, and even reverence, that geometers held for Euclid, the fifth postulate was regarded as a scandalous defect in his otherwise nearly perfect work. This feeling was not new in the eighteenth century. Even in ancient times geometers were dissatisfied with the fifth postulate, and it is likely that even Euclid himself was not entirely happy with it. He avoided using it in the proofs of his first twenty-eight theorems, but sheer necessity finally compelled its use in order to derive the full content of geometry as he wanted it. Through the centuries numerous attempts were made by various mathematicians to perfect Euclid's noble work by showing that the fifth postulate could in fact be proven from the other four, or failing that, to derive if from the other four together with another simple and self-evident postulate. By the early eighteenth century interest in such attempts was running high. By way of example we will look at just one of the most notable of them.

Girolamo Saccheri (1667–1733) was a Jesuit professor of theology, philosophy, and logic who had published a book *Logica Demonstrativa*, in which he wrote about the use of the method of logic known as *reductio ad absurdum*—a method related to proof by contradiction in mathematics. Saccheri became fascinated with Euclid's work and studied it thoroughly. He was well versed in the attempts of others to remove the "blemish" from *Elements*, and he had found the errors that defeated some of them. Thinking that he could succeed where others had failed, in 1733, while he was a professor at the University of Pavia, he published a book in which he applied *reductio ad absurdum* to the long-standing problem of the fifth postulate. The title of his book, which was written in Latin of course, would translate into English as something like *Euclid Freed of Every Flaw.*

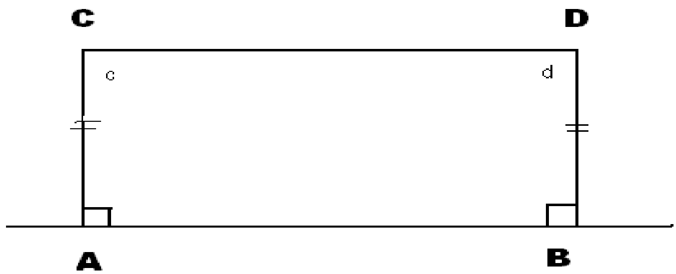
Saccheri's goal was to show that the fifth postulate was actually a theorem that could be proved from the other four postulates. His strategy in using *reductio ad absurdum* was to make the assumption that the fifth postulate was *false.* If he could then show, using the other four postulates, that this assumption led to a contradiction or "absurdity," it would follow that the *assumption* was false and the postulate must therefore be true. The way he chose to execute this strategy was by means of a rectangular figure as shown below, a figure known in his honor as a Saccheri quadrilateral, which I will refer to as an SQ in what follows. This geometrical construction consists of a base line AB on which two perpendicular segments of equal length AC and BD have been constructed. Using the first four postulates, Saccheri could show that the angles at C and D were equal to each other, so there were three possible cases:

Case 1: The angles C and D are both right angles.

Case 2: They are both obtuse, that is greater than right angles.

Case 3: They are both acute, that is less than right angles.

He could also prove that whichever of these cases held for *one* SQ would hold for *all* SQs, and that Case 1 was logically equivalent to the fifth postulate—the statement that he hoped to prove by his method of contradiction.



So his strategy of denying the fifth postulate amounts to the assumption that either Case 2 or Case 3 is always valid. He then need only show that each of these cases leads logically to a contradiction. With Case 2 he was indeed led to a contradiction, but only because he used the tacit assumption, not strictly justified by the first four postulates, that straight lines are infinite in length. He does not deserve too much criticism for this infraction because many other mathematicians, including Euclid himself, did the same thing.

In dealing with Case 3, Saccheri was faced with a more formidable task. He filled many pages with careful reasoning establishing strange proposition after strange proposition, apparently confident that he was closing in on an actual contradiction. He never did come to an actual contradiction, but he finally brought his labors (and his book) to an end by making a feeble and unjustified claim that his final propositions involving hazy notions about infinity contradicted the nature of lines. Thus ended the book in which he claimed to have proved, by *reduction ad absurdum*, that the fifth postulate is a consequence of the other four and could be properly placed among the theorems where it belonged.

As noted before, Saccheri was just one of a great many mathematicians who attempted to prove the fifth postulate. An indication of how widespread this effort was is provided by the fact that in 1763 a German graduate student by the name of G.S. Kluegel wrote a dissertation in which he carefully examined thirty such claimed proofs only to conclude that each and every one was defective.

Saccheri's work appears to have failed to convince the mathematicians of his day, for it was soon forgotten. It was not until more than a century and half later, long after it had been rendered obsolete by the work of others, that it was rediscovered and again brought to the attention of the mathematical community. By this time it had long been shown that the fifth postulate is in fact *independent* of the other four: neither it nor its negation contradict those first four postulates. Moreover, by this time a whole new field of non-Euclidean geometry had been developed using, in place of the fifth postulate, its negation. The great irony in this story is that the strange propositions that Saccheri had proved in his 1733 book were valid theorems in the new kind of geometry. If he had only proclaimed them as such, instead of claiming that some of them contradicted the nature of lines, he would today be regarded as the very first non-Euclidean geometer. He had actually been doing non-Euclidean geometry, but because of some mental block that he shared with all the mathematicians of his day, he was just unable to realize what it was that he had done. Could his lack of recognition of the nature of his own work have been caused by obstructions in the spiritual realm, obstructions that were soon to be removed in the coming reorganization of 1757? Discoveries made in the next several decades suggest that such could well be the case.

In 1829 there appeared a paper that is today regarded as the first publication in the field of non-Euclidean geometry. It was written by Nicolai Lobachevsky, a professor at the remote University of Kazan in Russia. Since it was written in Russian and was published in a very obscure journal, it attracted almost no attention among mathematicians or, of course, anyone else. But Lobachevsky was not the only person thinking along non-Euclidean lines. As often seems to be the case, when the world is ready for an idea, it tends to appear in the minds of more than one individual. In this case, one of the other minds was that of Janos Bolyai, a Hungarian army officer who was also a mathematician. His thoughts pertaining to the possibility of a real, authentic non-Euclidean geometry began as early as 1823, but it was not until 1829 (the year of Lobachevsky's publication) that he managed to get a manuscript of his results ready to be considered for publication. He submitted this manuscript to his own father, Farkas Bolyai, who was also a mathematician. Farkas was in the process of preparing a book of his own, on a different subject of course, for publication. So as it turned out, Janos Bolyai's work on non-Euclidean geometry appeared in 1832 as a twenty-six-page appendix to his father's book on another subject. Again it was not a publication destined to attract a lot of immediate attention. But from these humble beginnings in the obscure publications of Bolyai and Lobachevsky the subject of non-Euclidean geometry finally saw the light of day. At the time it caused no big stir. The authors were not well known, and the subject seemed too fantastic to be of real interest to serious mathematicians. But in the decades to come this new kind of geometry was to grow into a major intellectual revolution contributing to what I have called the liberation of mathematics. The consequences of this revolution, which are far too numerous to discuss in this paper, are being felt more and more as we proceed into the twenty-first century. As a single example that does indicate the nature of some of these consequences I will mention Einstein's theories of special and general relativity, which depend for their expression on mathematics resulting from non-Euclidean concepts.

It may seem that the period from the last judgment in 1757 until 1823, when Bolyai is first known to have been thinking in a non-Euclidean way, a span of sixty-six years, is an unduly long time for these liberating ideas to have been delayed. As a matter of fact, it actually took considerably less than sixty-six years for the revolutionary ideas to appear in the natural world. It is known that they were actively engaged in the mind of Carl Friedrich Gauss as early as 1792, reducing the time interval since 1757 to just thirty-five years.

Gauss (1777–1855) was for most of his life a professor at the University of Goettingen in Germany, but long before he attained a professorship he was a mathematical prodigy. He was a rare example of a child prodigy who not only retained his prodigious mental ability but extended it throughout a relatively long life. His mathematical achievements were such as to make him a towering giant in the whole history of world mathematics, surpassing even Euler in this regard. He is known to have been thinking, even while still a teenager, along the lines of creating a geometry in which the fifth postulate would not be true. But Gauss had one great weakness, if that is what it could be called. He hated criticism and controversy, so he was very hesitant to publish anything that could result in either. In his day the opinions of philosophers were extremely important in the universities of Europe, and with the influential philosophers it was considered definite that Euclidean geometry was an innate part of the human mind. Thus it would be preposterous for anyone, even a respected mathematician, to suggest that there could be a valid geometry in which some of the long-established theorems of Euclid failed to be true. So Gauss, not wanting to be subjected to scathing criticism or involved in bitter and prolonged controversy, declined to publish anything about his revolutionary thoughts on geometry. His aversion to those ordeals resulted in an unfortunate delay in the birth of non-Euclidean geometry. Not only would his publication initiating the new geometry have been decades earlier than those of Lobachevsky and Bolyai, but it would almost certainly have been better organized, and it definitely would have caught the attention and respect of a far wider circle of mathematicians. But, as we have seen, non-Euclidean geometry was nevertheless finally launched and went on to change the history of mathematics, science, and the world.

Algebra

Now I will consider the situation in the field of algebra in the eighteenth century and will show that here too there existed a very difficult problem that had stubbornly resisted all attempts to solve it. This problem pertained to finding the solutions of polynomial equations, that is, equations of the form

$$Ax^4 + Bx^3 + Cx^2 + \mathrm{D}\mathrm{x} + \mathrm{E} = 0$$

The *degree* of a polynomial is the highest exponent appearing, in this case 4, so this polynomial equation is said to be of fourth degree or a *quartic* equation. The simplest case is that of a first degree, or *linear*, equation: Ax + B = 0. Its solution is easily seen to be x = -B/A. Even second degree, or *quadratic*, equations, those that look like

$$Ax^2 + Bx + C = O$$

are relatively easy to solve, and beginning algebra students learn that the two solutions (the number of solutions being equal to the degree of the equation) are given by

$$x = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \text{ and } x = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

Quadratic equations could be solved even in ancient times, but the convenient algebraic notation that we take for granted today was not available then. Solving a quadratic in those days would not have been a task for a beginning math student.

When we come to the consideration of third degree or *cubic* equations, the subject becomes decidedly more challenging. Before proceeding further, note that in any polynomial equation you can always make the leading coefficient (that is, the coefficient of the highest power of x) equal to 1. If it is not already 1 to start with, you can merely divide both sides of the equation by whatever it happens to be, and the result will be an equivalent equation whose leading coefficient *is* 1. Thus you can think of the general cubic equation as being of the form

$$x^3 + Px^2 + Qx + R = 0$$

It was a major struggle for mathematicians of the Renaissance to come up with an algebraic formula, something like the two expressions above for the case of quadratics, that would provide the solutions for the cubic equation. Part of the difficulty, but only part, was due to the appearance of supposedly non-real entities such as square roots of negative numbers. A fairly easy step, however, was the discovery that the equation could always be modified by means of a simple substitution to obtain an equivalent equation in which there was no second degree term. So attempts to find the solutions could be concentrated on the so-called "depressed cubic equations" of the form

$$x^3 + px + q = 0$$

The story of the competition among sixteenth century mathematicians to find an algebraic expression for the solutions of this simple-looking equation is a fascinating episode in the history of mathematics. Interesting as they are, the details of this story will be omitted here because they are not really relevant to the point of this paper. I will be content to report that such an expression was finally found. There are of course three solutions since this is a third degree equation. Just to give an idea of what the long-sought answer looked like, here is one (the simplest) of the three:

$$x = \sqrt[3]{\frac{-q + \sqrt{q^2 + (4p^3/27)}}{2}} + \sqrt[3]{\frac{-q - \sqrt{q^2 + (4p^3/27)}}{2}}$$

Considering the great difficulties that were encountered in finding the algebraic solutions of the general cubic equation, we might expect that the general quartic equation, the one shown at the beginning of this section, would involve even more strenuous and prolonged efforts to solve. Surprisingly, this is not the case. It turns out that quartic equations can be reduced to cubic ones, so the results already obtained for the latter served to apply also to the former. Nevertheless, the final expressions for the four solutions of the general fourth degree equation are considerably longer and more complicated to write than the ones for the third degree case. There is, thankfully, no need to display any of them here. I will merely report that in 1545 solutions for both cubic and quartic equations were published in a book called *Ars Magna* by a colorful Italian mathematician named Girolamo Cardano (1501–1576).

Thus, by the middle of the sixteenth century, with the general cubic and quartic equations satisfactorily disposed of, mathematicians could turn their attention to the next one, the general equation of fifth degree, also known at the quintic equation:

$$Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F = 0$$

This equation does not appear to be very much more difficult than the quartic. Surely in a few years, perhaps a decade, algebraic expressions for its solutions would be found. (Let me emphasize here that by *algebraic* in this context we mean expressions involving only the four arithmetic operations of addition, subtraction, multiplication and division as well as the extraction of roots of all orders.) Mathematicians made an enthusiastic assault on the problem. In fact they made many such assaults using every kind of algebraic ingenuity they could imagine.

Omitting all the details of two centuries of mathematical history, I will simply note that over two centuries later, right up to 1757 and beyond, their heroic efforts had gotten them nowhere near the desired goal. Even the exalted Euler himself made more than one serious attempt at solving the frustrating problem, but his efforts too were unsuccessful. As late as 1762 he published a paper "On the Solution of Equations of Arbitrary Degree" in which he suggested a kind of canonical form that all solutions of such equations might fit. Although he did not determine any actual solutions, the fact that his canonical form was clearly an algebraic expression makes it obvious that he was of the opinion that general polynomial equations of every degree have solutions of the kind that mathematicians had been seeking for over 215 years. As it eventually turned out, he was wrong in that opinion.

It seems to have been almost the end of the eighteenth century, some four decades after the year of the last judgment, that a light began to dawn in the minds of a few mathematicians. A few were finally beginning to suspect that the big problem posed by the general quintic equation did not have a solution. In other words, that there is no algebraic expression which will provide solutions for that equation. Among these few there was a little-known Italian algebraist by the name of Paolo Ruffini (1765–1822), and there was a soon-to-be renowned mathematician named Carl Friedrich Gauss. In the case of Gauss, when he was still doing his graduate studies in the year 1799, he expressed in his dissertation the negative opinion that turned out to be correct. However, he offered nothing by way of proof.

Ruffini was apparently even earlier than Gauss in arriving at that opinion. He not only expressed the opinion, he even claimed to have a proof of it, which he published in that same year of 1799 in which Gauss gave the opinion only with no supporting evidence. Of course Gauss was not one to give "supporting evidence." He was not likely to publish any evidence short of a water-tight proof. Unfortunately for Ruffini, his proof was not convincing to his fellow mathematicians, and it was not even taken very seriously. In later years he published revised and presumably strengthened versions of his proof, but they met the same fate as his first.

It was not until 1826 that the problem of the quintic equation was finally settled once and for all. In that year a young Norwegian mathematician by the name of Niels Henrik Abel (1802–1829) published a complete and convincing proof that there exists no algebraic expression for the solution of the general quintic equation. It was published in the very first issue of the new mathematical periodical called the *Journal of Pure and Applied Mathematics*, which is (amazingly) still being published today over 180 years later. Abel's proof was an excellent example of the method of *reductio ad absurdum* that Saccheri had applied in geometry nearly a century earlier. An interesting sidelight is the tragedy that accompanied Abel's life, the extreme poverty and hardship that he endured while pursuing his mathematical studies (which extended far beyond polynomial equations), and his death from tuberculosis before reaching his twenty-seventh birthday.

Nearly seventy years had passed since the year 1757 that Swedenborg had associated with the last judgment. Is it reasonable to think that it would have taken that much time for spiritual enlightenment to have filtered down to the understanding of such a natural-world subject as polynomial equations? I think the answer to this question is a definite "yes" when we consider the slow but necessary advances that were occurring along the way. We have already mentioned the work of Ruffini in 1799, but there were even earlier ideas whose appearance contributed to the final result even though their authors were unaware of the negative conclusion they were leading to. As early as 1771 Joseph-Louis Lagrange (1736–1813) published a paper "Reflections on the Algebraic Solution of Equations" in which he applied the concept of permutations of all the solutions of an equation—a key concept in Abel's proof. Also, the work of Ruffini, which was done at least a quarter century before Abel's, came to be regarded as considerably more valid and worthy of recognition than had previously been the case. In recent years some historians of mathematics, in a belated attempt to accord Ruffini some well-deserved credit, are referring to Abel's result as the Abel-Ruffini Theorem.

The negative result for the problem of the quintic equation, as we can see now, was the end of an era in the history of mathematics, but at the time it did not cause much excitement. Recall that the same was true with regard to the negative outcome pertaining to the fifth postulate. Part of the reason for this is that, as mentioned, the active and exciting part of mathematics at the time, the part where new advances were being made with increasing rapidity, was analysis in which the amazingly fruitful concepts of calculus were still being exploited for comparatively easy results. For most mathematicians there was no need to struggle with centuries-old abstract problems that carried no likelihood of practical application anyway. But algebra was about to undergo a metamorphosis that would greatly expand its significance in the world of mathematics.

To see just a bit of the manner in which algebra was destined to change its role, we need to proceed a little further into the nineteenth century. Although Abel's paper (and perhaps even Ruffini's earlier ones) took care of the *general* quintic equation by showing that it had no algebraic solution, there are still certain *specific* quintic equations (as well as specific equations of even higher degree) that do have algebraic solutions. So there was still a challenging problem facing algebraists: how can you distinguish those polynomial equations that do have algebraic solutions from the ones that do not? Finding the answer to this question led to the discovery (or invention, depending on your philosophical attitude) of those algebraic structures that would today be called permutation groups. It was the work of a most unusual and colorful young mathematician named Evariste Galois.

Galois was born in 1811 and was killed in a duel with pistols before reaching his twenty-first birthday in 1832. Although we are interested primarily in his contribution to mathematics, his life was so extraordinary in its brevity and tragic ending that I must take the time and space for a few sentences of biographical comment. This unusual young French genius lived during times that were extremely tumultuous in the political history of his country. His strongly held and loudly proclaimed republican views kept him in a state of extreme stress much of the time, including even periods of incarceration. Moreover, he was also continually frustrated because his amazing mathematical talent, which in hindsight is quite apparent, was not often recognized during his lifetime. This may have been due in part to his uncompromising and somewhat belligerent personality, unrestrained by polite tactfulness. The precise circumstances leading to the duel are not known with any certainly, but a common account has it that his offence occurred at a boisterous public political gathering. It is said that he proposed a toast which was interpreted by some as a threat on the life of the sitting monarch. He was consequently challenged to the duel by a monarchist. In keeping with nineteenth century standards of honor, he could not decline, and so he accepted the challenge being quite certain that he would not survive. Apparently it is not even known who his opponent was.

To turn, more happily, to his mathematics, recall that the Abel-Ruffini result left many loose ends. Although it definitely established the fact that the *general* quintic equation had no algebraic solution, it said nothing about the conditions under which certain *specific* equations of degree five and higher *would* have such solutions. This is where Galois's work came in. His results make it possible to determine for any polynomial equation whether or not an algebraic solution exists. But this outcome in itself is not what makes his contribution important to the history of mathematics. The importance of his work lies in the algebraic superstructure that he built in order to obtain the result. By taking the idea of permuting the set of all solutions, which had been used by Ruffini, Lagrange, Abel, and others, and by regarding the collection of all possible permutations of solutions as a set whose elements could be combined (something like numbers can be multiplied together), he pushed these concepts to a new level and established the rudiments of what is today known as group theory. Galois showed that any specific polynomial equation of whatever degree would determine its own permutation group (as it would be called today) whose properties would depend on the coefficients in the polynomial. These properties would also provide the answer to the question of whether or not the equation had an algebraic solution.

Galois' work was destined to change the nature of mathematics, but that was not at all apparent to the mathematicians of 1832. In fact this work remained unknown to all but a handful of people for many years after his death. It was a decade before a mathematician of some reputation would take any interest it. That was another Frenchman, Joseph Liouville (1809–1882), who finally recognized the importance of what Galois had done and announced his result to the French Academy in 1843. In 1846 he published all of Galois' work in a new journal that he had started himself. From this point on, with ever increasing rapidity, algebra became a whole new game. It became an enterprise in which completely new abstract structures could be invented as needed—or desired.

As an indication of what is meant by an abstract structure, we will consider the concept of a group as it is regarded today. A group G is a collection of "elements" which can be "combined" with each other in some way. We will use a, b, c, etc. to denote the elements and + to denote the action of combining. These four axioms are required to be satisfied:

- Closure. The result of combining two elements in the collection is always another element of the collection. That is, a+b is always in G.
- Associativity. For all elements a, b, c a+(b+c) = (a+b)+c.
- There is an identity element i such that i+a = a for all a.
- Every element has an inverse. For each a there is some b such that a+b = i.

From these remarkably simple axioms comes forth the branch of modern mathematics known as group theory. In this way an algebraic structure has properties that are determined by a set of axioms in a manner that reminds us of geometry as done by Euclid already in ancient times.

But the group concept was just the first of many algebraic structures to be devised during the nineteenth and, even more so, the twentieth centuries. Algebra became a field for the invention of new entities, just as geometry, after the first sight of its non-Euclidean version, became a field where all manner of new and exotic geometries could flourish. Some of the major new algebraic structures, besides groups, that came into being in the nineteenth century are fields, rings, and vector spaces, but there are many others.

Geometry and Algebra Together

Let us now review briefly the similarities that existed in the development of geometry and algebra in the decades following 1757.

In each case there had been a long-standing problem that seemed frustratingly difficult. For geometry, the problem was that of proving the fifth postulate, which was complicated and lacking in self-evidence, from the other four which had the simplicity and obvious validity that was expected of postulates. For algebra, it was the problem of finding an algebraic solution for the general quintic equation, a seemingly reasonable next step following the solution of the general cubic and quartic equations some two hundred years previously.

There was a very gradual dawning, only *after* 1757, that these problems might in fact be impossible to solve. In the case of geometry, the earliest *known* inkling of this impossibility occurred to the brilliant young Gauss while he was still a teenager during the 1790s. In the case of algebra, the first *inkling* must also have occurred in the same decade, but in this case to two people: Ruffini, who actually published in 1799 what he thought was a proof of the non-existence of the quintic solution, and Gauss (again) who in his dissertation of the same year expressed his opinion that there was no solution but offered no proof.

The general acceptance, even among mathematicians, that these two famous problems had no solutions came about very slowly. For geometry it was not until the end of the 1830s that the non-Euclidean work of Lobachevsky and Bolyai gained credence, and even then there was no appreciation of the momentous consequences their ideas would have. For algebra it was the paper by Abel in 1826 that finally convinced most mathematicians that the search for the solution to the quintic was futile. (However, for another decade papers were still being presented claiming to have found the long-sought solution. See Derbyshire, p. 130.)

For both geometry and algebra an era had obviously ended. No need to persist in the attempt to do the impossible. But also for each, a new and far more glorious era was beginning, arising in both cases with the help of ideas that had been developed in the assault on those impossible problems. In one case, this meant the creation of new geometries in which the valid theorems were the same as the seemingly preposterous conclusions the former geometers such as Saccheri had deduced in their attempts at *reductio ad absurdum.* In the other case, it meant using the concept of permutations of the solutions of a polynomial to create a new algebraic structure which came to be called a group, the first of many to come.

The new era that opened for geometry and algebra during the early decades of the nineteenth century can be regarded as a liberation of these subjects. In the new era they were no longer bound to the physical world and to the so-called real numbers that had been used to quantify it. Henceforth mathematics would be open to all manner of newly devised structures that could be studied for their innate beauty as well as their ability to model physical reality. This freedom brought with it a stronger commitment to logical rigor in the proof of accepted theorems. Mathematical proofs could no longer make use of subtle tacit assumptions derived from the senses like the older proofs did, even those of the revered Euclid.

Mathematics in the New Era

The revolutionary nature of the transformations that had occurred in geometry and algebra was not immediately recognized in either case. It was not until 1846 that Galois's achievement was given due respect by mainstream mathematicians, and even then its recognition was due primarily to the efforts of one man: Liouville, as already noted. In the case of geometry, it was Bernhard Riemann (1826–66) who first clearly saw the amazing possibilities that were implicit in geometry freed from its Euclidean mold. By applying the methods of calculus to the study of geometry itself, he extended the work of Gauss in the field of differential geometry, making possible all sorts of new geometries with any number of dimensions and with curvature that could vary with location. His famous lecture "On the Hypotheses Which Lie at the Foundations of Geometry," given at Goettingen, Germany, in 1854, set the stage for the theory of relativity and the field of modern cosmology.

I think it would be fair to say that by this year of 1854 the liberation of geometry and algebra had spread throughout mathematics as a whole. Yes, nearly a century had passed since 1757, but Swedenborg *had* foretold that the effects of the last judgment would only slowly appear in the natural world. From this time forward mathematics gathered ever-increasing momentum. In this paper I can give only a slight hint about just a few of the developments.

The English mathematician Arthur Cayley (1821–95) took up the study of abstract groups, creating a well-organized framework for explaining and extending Galois's brilliant insights. The resulting field of group theory is still growing in content and applications. The so-called imaginary numbers (square roots of negative numbers), which had been lurking around the fringes of algebra for many decades and were regarded with deep suspicion by respectable mathematicians, finally were welcomed into the mainstream of mathematics and even accorded a position of honor.

Sir William Rowan Hamilton (1805–65) was so impressed with the two-dimensional complex number system (formed from the real and imaginary numbers together) that he struggled mightily to construct a three-dimensional number system. When that goal proved to be impossible, he founded instead a four-dimensional system of "numbers" that he called quaternions. These entities, strange as they were, never had to endure the ostracism that was suffered in earlier times by their comparatively ordinary cousins, the imaginary numbers.

Late in the nineteenth century the study of analysis situs, now called topology, came into being, the ultimate level of abstraction in dealing with the concept of space. The eminent French mathematician Henri Poincaré (1854–1912) applied algebraic structures to the study of topological spaces, thereby creating algebraic topology. Geometry and algebra became more and more freely combined to the benefit of all mathematics, a process that continued throughout the twentieth century. Such subjects as algebraic geometry, algebraic number theory, topological groups, combinatorial geometry, and many others developed into important parts of the mathematical enterprise.

Sir Michael Atiyah, a notable contemporary English mathematician, in his 2001 survey of twenthieth-century mathematics remarks on the ancient mathematical dichotomy between geometry and algebra, and he calls them "the two formal pillars of mathematics" (Atiyah 2001, 657). Their primary importance in the structure of mathematics is more clearly evident today than it was in the middle of the eighteenth century, when they had been temporarily placed on a back burner. In hindsight it all seems quite natural that their hard-won emancipation from the tight bonds of tradition should in due course have led to the liberation currently enjoyed by all mathematics. The timing of those first halting steps toward freedom, coming as they did in the final decades of the eighteenth century, suggests to me that they were consequences of the spiritual reorganization that Swedenborg identified with the last judgment. ❑

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