MORE ON MATRICES

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Professor Edward F. Allen's introduction of the matrix as a conceptual device in his article, "God, Man, and World,"[1] brought to mind some related applications of the matrix to non-mathematical ideas. Of special interest is the use of the matrix as a concrete representation of a transformation. Perhaps one could represent the transformation of man from evil to good by a 'regeneration' matrix. But before developing this thought further it may be useful to recall the rules for matrix multiplication.

We will limit ourselves to a single or double column and double row matrices. Let the matrix $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ represent the two–component state of a system. (The use of 'system' here may apply to a mathematical situation, a political situation or even a psychological state.) The matrix $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{22} & \alpha_{22} \end{pmatrix}$ will be used to transform the state $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ to a new state $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Symbolically the process is written as:

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

If the quantities in the matrices shown are numbers or some more general algebraic form, the multiplication process is such that:

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \alpha_{11}a_1 + \alpha_{12}a_2 \\ \alpha_{21}a_1 + \alpha_{22}a_2 \end{pmatrix},$$

or $b_1 = \alpha_{11}a_1 + \alpha_{12}a_2$ and $b_2 = \alpha_{21}a_1 + \alpha_{22}a_2$. In other words the new state is expressed in terms of the old state and the elements, α_{ij}, of the transforming matrix. Let us sharpen our skills with a numerical example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a + 2b \\ 3a + 4b \end{pmatrix}.$$

[1] THE NEW PHILOSOPHY, July–Sept., 1976, pp. 441–459.

There are many special matrices which do special tasks. For example, the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ leaves any state unchanged

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

but the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ will interchange the components of $\begin{pmatrix} a \\ b \end{pmatrix}$ as follows:

$$\begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Usually a transformation matrix has an inverse transformation matrix which will undo the work of the original transformation. For example:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$$

but

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

and one may deduce that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which we know causes no transformation at all.

With this little introduction to matrix multiplication, let us return to the idea of a 'regeneration' matrix. Suppose, in a simplistic way, that the state of man could be represented by a two component matrix, $\begin{pmatrix} \text{fraction of good} \\ \text{fraction of evil} \end{pmatrix}$. If a man were entirely good, his state would be $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and if entirely evil, his state would be $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Therefore a 'total' regeneration matrix would be one which transformed the state $\begin{pmatrix}0\\1\end{pmatrix}$ to the state $\begin{pmatrix}1\\0\end{pmatrix}$. A reasonable choice for the total regeneration matrix is therefore

$$\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \text{ since } \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}.$$

Similarly one could define a 'total' degeneration matrix which would transform $\begin{pmatrix}1\\0\end{pmatrix}$ to $\begin{pmatrix}0\\1\end{pmatrix}$ and such a suitable matrix might be $\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$.

Clearly our matrix scheme is very unrealistic. Mathematicians will note that the regeneration and degeneration matrices are not inverses of each other. Theologians will observe that man is never totally good or evil and that regeneration occurs in tiny increments; not in single leaps between pure and impure states. However, rather than reject the scheme entirely it may be useful to retain some parts. For example, the state of a young person entering adult life might be represented by the matrix $\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}$ with an equal opportunity of becoming good or evil. The regeneration matrix can be modified to take account of the gradual nature of the process. One possibility is the matrix **R** such that

$$\mathbf{R} = \begin{pmatrix}1 & r\\-r & 1\end{pmatrix}$$

where r is some small fraction (say .01). Then the matrix **R** acts upon the state $\begin{pmatrix}.5\\.5\end{pmatrix}$ to produce an improved state:

$$\begin{pmatrix}1 & .01\\-.01 & 1\end{pmatrix}\begin{pmatrix}.5\\.5\end{pmatrix} = \begin{pmatrix}.505\\.495\end{pmatrix}.$$

The fraction r could be a variable quantity depending on a particular life situation. A very deep regenerative experience might have a relatively high value for r. However, it is expected that r will always have a small absolute value. Assuming this is the case the matrix **R** has some interesting properties.

First there exists an approximate inverse

$$\mathbf{R}^{-1} = \begin{pmatrix}1 & -r\\r & 1\end{pmatrix} \text{ such that}$$

$$RR^{-1} = \begin{pmatrix} 1 & r \\ -r & 1 \end{pmatrix}\begin{pmatrix} 1 & -r \\ r & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

within a small factor of r^2. (Note that if $r = .01$ then $r^2 = .0001$ is very small.) Such an approximate inverse acts as a 'degenerative' matrix, a process which, like regeneration, is also expected to be gradual. (We recall that the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ leaves a state unchanged.)

Another property of **R** which should emerge is that if the transformation **R** is applied two times consecutively it should produce a result which is equivalent to doubling the value of r for a single application of **R**. Stated mathematically

$$\begin{pmatrix} 1 & r \\ -r & 1 \end{pmatrix}\begin{pmatrix} 1 & r \\ -r & 1 \end{pmatrix} \text{ should equal } \begin{pmatrix} 1 & 2r \\ -2r & 1 \end{pmatrix}.$$

By applying the multiplication rules we observe that $\mathbf{R}(r)\,\mathbf{R}(r)$ equals $\mathbf{R}(2r)$ except for some small terms in r^2. As an example of repeated application of **R** to the original state $\begin{pmatrix} .5 \\ .5 \end{pmatrix}$ the following calculations are given

$$\mathbf{R}\begin{pmatrix} .5 \\ .5 \end{pmatrix} = \begin{pmatrix} .505 \\ .495 \end{pmatrix}; \qquad \mathbf{R}^2\begin{pmatrix} .5 \\ .5 \end{pmatrix} = \begin{pmatrix} .5099 \\ .4899 \end{pmatrix};$$

$$\mathbf{R}^3\begin{pmatrix} .5 \\ .5 \end{pmatrix} = \begin{pmatrix} .5148 \\ .4848 \end{pmatrix}; \qquad \mathbf{R}^4\begin{pmatrix} .5 \\ .5 \end{pmatrix} = \begin{pmatrix} .5197 \\ .4797 \end{pmatrix}$$

and eventually

$$\mathbf{R}^{10}\begin{pmatrix} .5 \\ .5 \end{pmatrix} = \begin{pmatrix} .5477 \\ .4478 \end{pmatrix}.$$

We observe the steady increase in the upper 'good' component of man's state.[2]

Matrices are particular concrete representations of transformations; and we have suggested one possible mode. Other concrete forms might also be used to illustrate schema from areas other than mathematics. Does the differential operator have an ap-

[2] One must be careful not to apply the matrix **R** too often. Eventually the state takes on negative components. The mathematically inclined reader will note that **R** is similar to an infinitesimal rotation and therefore $\mathbf{R}^n$ causes inapplicable states for large n.

plication of the above type? Perhaps this initial attempt will encourage readers to explore this fruitful area.

The concept of a transformation is a general one and surely has application to areas of civic, social and religious life. While the use of matrices in these other areas may seem to be rather artificial and over-simplified, there is some satisfaction in observing the basic unity of thought patterns. Since the mind of man has a basic structure, it seems only reasonable that his theoretical concepts in the various disciplines such as mathematics and theology would exhibit corresponding patterns.

A CHANGE OF SECRETARY

In the fall of 1976, after eighteen years of dedicated service as Secretary of this Association, Professor Morna Hyatt found it necessary to resign from that office. Miss Hyatt had accepted growing responsibilities in administration and instruction in the Girls School of the Academy of the New Church, until she felt it necessary to give those uses her undivided attention.

Fortunately Miss Hilary Pitcairn had just returned to enter the use of teaching after completing her formal education at Oxford, and she accepted the appointment by the President as Secretary.

We extend our best wishes to both Morna Hyatt and Hilary Pitcairn in their respective uses.

E.F.A.